Mechanics Problems: Dynamics and Moment of Inertia

Rigid Bodies, Rotation, and Composite Laminas Inclined Plane, Semicircular Lamina, Composite Lamina, Rod Framework

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Before attempting the problems, please review the following background material:

- **Newton's Second Law:** $F = ma$ for translational motion
- **Rotational Dynamics:** $\tau = I\alpha$ for rotational motion
- **Moment of Inertia:** $I = \sum mr^2$ for point masses, $I = \int r^2 dm$ for continuous bodies
- **Parallel Axis Theorem:** $I = I_{\text{cm}} + Md^2$
- **Perpendicular Axis Theorem:** For planar bodies, $I_z = I_x + I_y$
- **Centre of Mass:** For a semicircular lamina, $\bar{y} = \frac{4a}{3\pi}$ from the diameter

No calculus is required unless explicitly stated. Students are encouraged to rely on:

- Free-body diagrams
- Symmetry arguments
- Thoughtful choice of reference frame
- Standard moment of inertia formulas

Conceptual Background

These problems explore the dynamics of connected objects on inclined planes and the calculation of moments of inertia for various composite shapes. The first problem combines translational and rotational motion with friction on an inclined plane. The remaining problems focus on determining moments of inertia for laminas and frameworks using symmetry, standard results, and the parallel axis theorem.

Key Definitions / Laws

Newton's second law for translation and rotation:

$$\sum F = ma, \quad \sum \tau = I\alpha$$

Moment of inertia formulas:

$$I_{\text{rod about centre}} = \frac{1}{12}ML^2, \quad I_{\text{rod about end}} = \frac{1}{3}ML^2$$

$$I_{\text{disk about centre}} = \frac{1}{2}MR^2, \quad I_{\text{semicircle about centre}} = \frac{1}{2}MR^2 \text{ (for full disk)}$$

Parallel axis theorem:

$$I = I_{\text{cm}} + Md^2$$

Perpendicular axis theorem (for planar bodies):

$$I_z = I_x + I_y$$

Centre of mass for a semicircular lamina:

$$\bar{y} = \frac{4a}{3\pi} \text{ from the diameter}$$

Problem 1. *Block and Cylinder on Inclined Plane.*

On an inclined plane of 30° a block, mass $m_2 = 4$ kg, is joined by a light cord to a solid cylinder, mass $m_1 = 8$ kg, radius $r = 5$ cm (Fig. 1). Find the acceleration if the bodies are released. The coefficient of friction between the block and the inclined plane $\mu = 0.2$. Friction at the bearing and rolling friction are negligible.

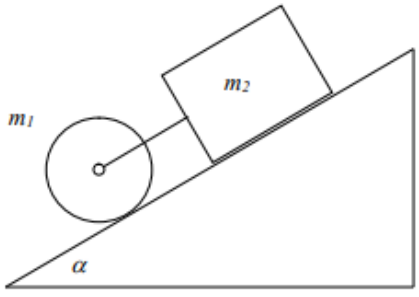

Figure 1

Figure 1: Block and cylinder connected by a light cord on an inclined plane of 30°.

Problem 2. *Semicircular Lamina Moment of Inertia.*

A uniform lamina has mass m and is in the shape of a semicircle of radius a, centred at the point O. The centre of mass of the lamina is at the point G. The lamina is free to rotate about a fixed smooth horizontal axis L, which is perpendicular to the plane of the lamina and passes through G. Calculate the moment of inertia of the lamina about L, in terms of m and a.

Problem 3. *Composite Lamina Moment of Inertia.*

A uniform circular lamina has radius $4a$ and centre O. The points A, B, C and D lie on the lamina and are vertices of a square whose centre is at O so that $OD = 2a$. Four circular discs, each of radius a, with centres A, B, C and D are removed from the lamina. The remaining lamina forms a new composite lamina of mass m. The new lamina is free to rotate in a vertical plane about a fixed smooth horizontal axis L, which is perpendicular to the lamina and passes through a point P at the circumference of the lamina. Calculate the moment of inertia of the lamina about L, in terms of m and a.

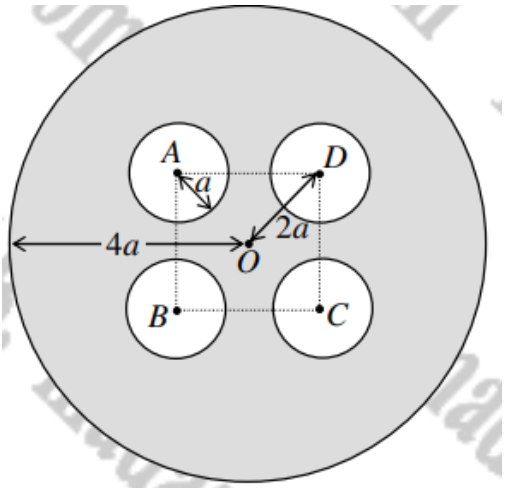

Figure 2: Circular lamina with four circular holes removed. Axis L passes through point P on the circumference.

Problem 4. *Rod Framework Moment of Inertia.*

Four identical rods, each of mass m and length $2a$ are joined together to form a square rigid framework $ABCD$. A fifth rod AC, of mass $3m$, is added to the framework for extra support. The 5 rod framework is free to rotate about an axis L, which passes through A, and is perpendicular to the plane of $ABCD$. Determine the moment of inertia of the framework about L.

Topics covered before this problem set

1. Newton's laws of motion
2. Rotational dynamics and torque
3. Moment of inertia for standard shapes
4. Centre of mass calculations

Topics covered in this problem set

1. Combined translational and rotational motion on inclined planes
2. Rolling without slipping constraints
3. Friction in rolling motion
4. Moment of inertia for semicircular laminas
5. Centre of mass for non-symmetric shapes
6. Parallel axis theorem applications
7. Composite shapes with holes
8. Framework of rods and diagonal supports
9. Geometric distance calculations for moment of inertia

Next up

1. Angular momentum conservation
2. Rotational kinetic energy
3. Gyroscopic motion
4. More complex composite bodies